Instanton for random advection

Michael Chertkov*
Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel
(Received 27 June 1996)

A path integral over trajectories of $2n$ fluid particles is identified with a $2n$th order correlation function of a passive scalar convected by d-dimensional short-correlated multiscale incompressible random velocity flow. Strong intermittency of the scalar is described by means of an instanton calculus (saddle point plus fluctuations about it) in the path integral at $n \gg d$. The anomalous scaling exponent of the $2n$th scalar's structural function is found analytically. [S1063-651X(97)00703-4]

PACS number(s): 47.10.+g, 47.27.−i, 05.40.+j

I. INTRODUCTION

The problem of scaling behavior in Kraichnan's model of a white-advected passive scalar [1] attracts a great deal of attention [2–7]. In the wide range of scales, called the convective interval, structural functions of the Lagrangian tracer θ, passively advected by d-dimensional short-correlated in time multiscale incompressible flow, possess a scaling behavior. The anomalous scaling exponent ζ_{2n} of the $2n$th order structural function, $\langle(\theta_1-\theta_2)^{2n}\rangle \propto r^{\zeta_{2n}}$, has been calculated in the following cases: (i) large space dimensionality $\zeta_2 d \gg (2-\zeta_2)n$, for $n=2$ in [4], and generally for all allowed n in [6], $\zeta_{2n} \to 2n(n-1)(2-\zeta_2)/d$; (ii) almost smooth scalar field $2-\zeta_2 \ll 1$, $d>2$ for $n=2$ in [5] and generally for $n(2-\zeta_2) \ll d$, $d>n$ in [7], $\zeta_{2n} \to 2n(n-1)(2-\zeta_2)/(d+2)$. The perturbation methods yield the scaling exponents in the limits where the respective bare approximations are strictly Gaussian and the anomalous corrections are small.

Instanton (steepest-descent) formalism after perturbation expansion is the second quantitative method that could be applied to a general statistical problem. The method works when a large parameter causes some very special rare configuration to have an exponentially large weight. Such a large parameter may be a high order n of correlation function $\langle\varphi^n\rangle$ of fluctuating field φ. The bare instanton approximation is obviously strongly non-Gaussian. The idea was originally introduced and successfully applied in field theory [8] almost 20 years ago, but introduced to the turbulence theory only very recently. An instanton calculus in a Lagrangian path integral was used to find an exponential tail of the scalar's probability distribution function (reflected intermittent, non-Gaussian behavior of higher moments) in the case $\zeta_2=0$ of linear velocity profile [9] (later on it was shown that the limit turns out to be solvable exactly [10–12]). A general method for finding the non-Gaussian tails of probability distribution functions (PDF) for solutions of a stochastic differential equation, such as the convection equation for a passive scalar, random driven Navier-Stokes, etc., was formulated in [13]. The initial idea of the method is to look for a saddle-point configuration in the path integral for the generating functional introduced in [14,15]. The extremum of the effective action is given by a coupled field-force configuration (instanton), varying in space and time. The method was applied recently to Burgers's turbulence [16,17]. Generally, it is very difficult to solve the coupled (field-force) instanton equations.

In the present paper we generalize the idea of [9] for the case of a nonlinear velocity profile ($\zeta_2>0$). ζ_{2n} is calculated for n being the largest number in the problem. The method is based on a very special feature of the problem [9]: there exists a closed differential equation connecting $2n$th and $(2n-2)$th simultaneous correlation functions of the scalar. The $2n$th correlation function is expressed via the convolution of the resolvent of the eddy-diffusivity operator with a source function constructed from the $(2n-2)$th correlation function. To prepare a path integral for the instanton calculus, we perform an explicit map of the original problem of calculation of the $2n$th order correlation function to the problem of calculation of a matrix element in an auxiliary $2n$-particle quantum mechanics. The resolvent of the eddy-diffusivity operator is expressed in the method via the path integral over trajectories of $2n$ fluid particles moving from an initial geometry (at which we are aimed to describe the scalar's correlations) with a characteristic scale r to a final large-scale ($\sim L$) geometry. The tensor of eddy diffusivity plays the role of tensor of inverse mass for the particles from the associated quantum mechanics. The tensor depends explicitly on relative distances between the particles.

It is the large number of particles that makes the auxiliary quantum mechanics almost ''classical'' (''semiclassical''). A classical $2n$-particle configuration is the desirable rare event that describes both the intermittency of $2n$th moments of scalar differences and intermittency of the nth moment of the dissipation field $\varepsilon=\kappa(\nabla\theta)^2$ [it is proven in [6] that they are related to each other, if scale invariance of the structural function and of the correlation function of the dissipation field is valid: $\langle\varepsilon^n\rangle \sim (L/r_d)^{\Delta_{2n}}$, $\Delta_{2n}=n\zeta_2-\zeta_{2n}$; κ and r_d are the diffusion coefficient and scale, respectively]. Calculation of the ''classical'' (saddle-point) contribution into $\langle\varepsilon^n\rangle$ gives the scale-invariant answer: $\Delta_{2n}^{\rm cl} \to n\zeta_2$ at $n\to\infty$. The classical anomalous behavior shows the highest level of intermittency possible: $\zeta_\infty^{\rm cl}=0$. Therewith exist a wide set of classical trajectories (realizing themselves separately, for different initial displacements of the points and different forms of the sca-

*Present address: Department of Physics, Princeton University, Princeton, NJ 08544.

lar's source) responsible for the classical answer. To extract an optimal trajectory (that gives the lowest possible contribution of fluctuations) from the set of the classical ones and thus to get a finite asymptotic for ζ_{2n}, we must account for the fluctuations about the saddle points.

It is shown that, at $0<\zeta_2<1$, the optimal trajectory is defined as a relative dispersion of two groups (drops) of particles: there is one distance (separation between the drops) being stretched while all the other distances (sizes of the drops) are being contracted dynamically. They are Gaussian fluctuations about the optimal trajectory that should give the true value of ζ_{2n}. Accounting for the relative longitudinal (along the ''classical'' stretching direction) Gaussian fluctuations of the drops gives the dominant contribution into the n-independent asymptotic for ζ_{2n} at $n\to\infty$. The exponent is finite, it grows linearly with d and decreases monotonically with an increase in ζ_2. The finite limit for ζ_{2n} at $\zeta_2\to 0$ along with the non-anomalous answer $\zeta_{2n}=0$ for the strictly ''logarithmic'' limit $\zeta_2=0$ [9,10] show together a discontinuity of ζ_{2n} at $\zeta_2=0$. There exists a simple physical picture that explains the origin of this discontinuity. In the first case of a linear velocity profile, distances between all the fluid particles are stretched by linear diffeomorphisms: there is no way for two groups of particles to diverge from each other and to keep the inner group distances contracted (or even intact) simultaneously. On the contrary, in the case of finite ζ_2 (yet ζ_2 should be smaller than unity), the two-point trajectory, with the sizes of the drops of particles being contracted dynamically whereas the distance between the drops being increased, is allowed.

Most non-Gaussian fluctuations about the saddle-point configuration can be dropped in comparison with the Gaussian ones if $n\gg\mathrm{Pe}$ (Péclet number), d (we should only worry about the explicit calculation of the non-Gaussian fluctuations corresponding to a soft rotation mode). This method is not applicable for Pe being of the order of (the more so as being larger than) n. However, making use of an overall observation (concerning the linearity of the problem and the scale-invariance feature of different terms entered in the correlation functions) one can extend the anomalous result (but not the method used for its derivation) to the limit $\mathrm{Pe},n\gg d$ too.

The two-point configuration is not relevant at $1<\zeta_2<2$ (repulsion of particles inside of a drop is no longer weak to make the configuration stable dynamically). Only trajectories with many ($\sim n$) distances being diverged should be taken into account. However, calculation of fluctuations about such trajectories shows a strong renormalization of the saddle-point answer: it is a product of $\sim n$ algebraic terms (each responsible for fluctuation of a distance) that makes the contribution of fluctuations competitive with (or even larger than) the classical value. The resulting contribution to $\langle\varepsilon^n\rangle$ is negligible in comparison with the normal scaling term, that always exists. To conclude, the instanton calculus is not an appropriate tool in this case.

The material in this paper is organized as follows. In Sec. II, after a detailed and formal definition of the problem we introduce path integral representation for the $2n$th order correlation function of the passive scalar. We present the path integral for $\langle\varepsilon^n\rangle$ too. It completes preparation for delivering an instanton (steepest-descent) formalism for calculation of $\langle\varepsilon^n\rangle$ at $n\gg d$ in the two forthcoming sections. Saddle-point equations are derived and studied in Sec. III. The contributions of different saddle points to $\langle\varepsilon^n\rangle$ are calculated (Appendix A) and compared with each other in Sec. III B. To improve the saddle-point calculations and to extract among the saddle points an optimal one we study Gaussian fluctuations about the saddle points in Sec. IV and Appendixes B and C. The anomalous exponent ζ_{2n} for the optimal saddle point is calculated there in Sec. IV. In the concluding Sec. V we discuss the results from the points of view of criteria of their applicability, restrictions imposed, possible generalizations, and comparison with other results and methods.

II. FORMULATION OF THE PROBLEM

The advection of passive scalar is governed by the equation

$$(\partial_t+v_\alpha\nabla_\alpha-\kappa\Delta)\theta=f,\quad \nabla_\alpha v_\alpha=0, \tag{2.1}$$

where $f(t;\mathbf{r})$ is the external source, $\mathbf{v}(t;\mathbf{r})$ is the advecting d-dimensional velocity, and κ is the diffusion coefficient. $f(t;\mathbf{r})$ and $\mathbf{v}(t;\mathbf{r})$ are independent random functions of t and $\mathbf{r}$, both Gaussian and δ correlated in time. The source is spatially correlated on a scale of the pumping L, i.e., the pair correlation function $\langle f(t_1;\mathbf{r}_1)f(t_2;\mathbf{r}_2)\rangle=\delta(t_1-t_2)\chi(r_{12})$ as a function of its argument decays on the scale L. The value of $\chi(0)=P$ is the production rate of θ^2. The eddy-diffusivity tensor $\mathcal{K}^{\alpha\beta}$, which describes the Gaussian velocity correlations

$$\langle v^\alpha(t_1;\mathbf{r}_1)v^\beta(t_2;\mathbf{r}_2)\rangle=\delta(t_1-t_2)[V_0\delta^{\alpha\beta}-\mathcal{K}^{\alpha\beta}(\mathbf{r}_1-\mathbf{r}_2)], \tag{2.2}$$

$$\mathcal{K}^{\alpha\beta}(\mathbf{r})=\frac{D}{(2-\gamma)r^\gamma}[(d+1-\gamma)\delta^{\alpha\beta}r^2-(2-\gamma)r^\alpha r^\beta], \tag{2.3}$$

depends on two parameters: D, which defines the level of turbulence, and γ, $0<\gamma<2$, which measures a degree of non-
smoothness of the velocity field.

Averaging Eq. (2.1) over the statistics of $\mathbf{u}(t;\mathbf{r})$ and $f(t;\mathbf{r})$, one gets the closed equation for the simultaneous correlation functions of the scalar $F_{1,\ldots,2n}=\langle\theta(\mathbf{r}_1),\ldots,\theta(\mathbf{r}_n)\rangle$ [9]:

$$-\hat{\mathcal{L}}_{2n}F_{1,\ldots,2n}=\chi_{1,\ldots,2n}, \tag{2.4}$$

$$\chi_{1,\ldots,2n}=\chi_{12}F_{3,\ldots,2n}+\text{permutations}, \tag{2.5}$$

$$\hat{\mathcal{L}}_n=-\sum_{i\neq j}^{n}\mathcal{K}^{\alpha\beta}(\mathbf{r}_i-\mathbf{r}_j)\nabla_i^\alpha\nabla_j^\beta+\kappa\sum_i^n\Delta_i. \tag{2.6}$$

The dependence of the source function $\chi_{2n}(r_{ij}\sim L)$ on L at $r_{ij}\lesssim L$ is estimated as $\sim L^{(n-1)\gamma}$; the function decays algebraically fast at the largest scales, $r\gg L$. It is the major information about χ_{2n} required for further consideration.

Equation (2.5) for the pair correlation function ($n=2$) was solved explicitly [1]. The pair correlation function in the convective interval, $r_d\ll r_{12}\ll L$, where $r_d^{2-\gamma}=2(2-\gamma)\kappa/[D(d-1)]$, gets the form

$$\langle\theta_1\theta_2\rangle=P\frac{2-\gamma}{\gamma(d-1)D}\left(\frac{L^\gamma}{d-\gamma}-\frac{r^\gamma}{d}\right). \tag{2.7}$$

Thus the pair structural function is shown to have a simple scaling behavior in the convective interval, $\langle(\theta_1-\theta_2)^2\rangle\sim Pr^{\gamma}/D$, $\zeta_2=\gamma$, to provide the constancy of the flux of θ^2 there. The scaling exponent of $\hat{\mathcal{L}}$ is $-\gamma$, the function χ_{12} does not depend on r_{12} deep inside the convective interval so that its exponent is 0, the solution of Eq. (2.7) thus may be presented in the form $F_{\rm forc}+\mathcal{Z}$, where we separated the so-called ''forced'' part of the solution (with the scaling exponent γ) from the zero mode (that is constant in this case). It is the forced part that contributes the second order structural function. The separation for ''forced'' terms and zero modes is valid for higher order correlation functions as well. It has been recognized independently by the authors of [4,5,18] that there are zero modes $\mathcal{Z}$ that may provide for an anomalous scaling. A zero mode, possessing the slowest down scale decrease among the ones built on $2n$ points (that is not reduced to a sum of zero modes each built on a less number of points), gives the major contribution into the $2n$th order structural function of the scalar, for $n>1$ [6]. Scaling of such a zero mode should grow with n to provide the convexity of ζ_{2n} as a function of n (it is an immediate consequence of the Holder inequality, see, for example, [19]). There are two Gaussian limits where there is no anomalous scaling and it is easy to make a classification of zero modes of operator $\hat{\mathcal{L}}$ there: (a) limit of large space dimensionality, $d=\infty$; (b) so-called ''diffusive'' limit of the smooth scalar field, $\gamma=2^-$ [to be precise it was done even in a more restrictive case, when $D/(2-\gamma)$ is finite]. It was a recent breakthrough in the analytic theory of turbulence, when the anomalous exponent ζ_{2n} was calculated perturbatively in the leading non-Gaussian order in the respective small parameters: $1/d$ in [4,6] and $2-\gamma$ in [5,7]. One emphasizes that both the perturbative techniques do not work for sufficiently large moments $2n$, when the anomalous corrections are of the order of the normal scaling exponent $n\gamma$. To deal with ζ_{2n} for the largest moments, we shall deliver a nonperturbative instanton technique.

The basic equation (2.4) can be rewritten in the following evolution form [one step back from the derivation of Eq. (2.4) presented in [4]]:

$$F_{1,\dots,2n}=\int_0^\infty dT\exp\left[T\left(-\frac{1}{2}\,\mathcal{K}_{ij}^{\alpha\beta}\nabla_i^\alpha\nabla_j^\beta+\kappa\sum_i\,\Delta_i\right)\right]\times\chi_{1,\dots,2n}=\int_0^\infty dT\int\prod_i\,d\mathbf{R}_i\mathcal{R}(T;\mathbf{r}_i,\mathbf{R}_i)\chi_{1,\dots,2n}(\mathbf{R}_i). \tag{2.8}$$

Here and everywhere below summation over the repeated particle and dimensional indexes will be assumed. $\mathcal{R}(T;\mathbf{r}_i,\mathbf{R}_i)$ is the resolvent of the operator $\hat{\mathcal{L}}_{2n}$,

$$(\partial_t-\hat{\mathcal{L}}_{2n}\{\mathbf{r}\})\mathcal{R}(t;\mathbf{r}_i,\mathbf{R}_i)=\delta(t)\prod_i\,\delta(\mathbf{r}_i-\mathbf{R}_i). \tag{2.9}$$

Considering the differential operator under the exponent from the first line of Eq. (2.8) as a Hamiltonian of a $2n$-particle quantum mechanics we can rewrite the resolvent in the Hamiltonian form of the standard Feynman-Kac path integral

$$\mathcal{R}(T;\mathbf{r},\mathbf{R})=\int_{\boldsymbol{\rho}_i(0)=\mathbf{r}_i}^{\boldsymbol{\rho}_i(T)=\mathbf{R}_i}\prod_i^{2n}\,\mathcal{D}\boldsymbol{\rho}_i(t)\mathcal{D}\mathbf{p}_i(t)\times\exp[-\mathcal{S}(\boldsymbol{\rho}(t);\mathbf{p}(t))], \tag{2.10}$$

$$\mathcal{S}=\int_0^T dt\left\{\frac{1}{2}p_i^\alpha[\mathcal{K}]_{ij}^{\alpha\beta}p_j^\beta-p_i^\alpha\dot{\rho}_i^\alpha\right\}, \tag{2.11}$$

where $[\hat{\mathcal{K}}]$ is defined as

$$[\mathcal{K}]_{ij}^{\alpha\beta}=\mathcal{K}^{\alpha\beta}(\boldsymbol{\rho}_i-\boldsymbol{\rho}_j)-2\kappa\delta^{\alpha\beta}\delta^{ij}. \tag{2.12}$$

Retarded regularization of the ''mass'' ($[\hat{\mathcal{K}}]$ term) in the action is considered, that means the following discretization procedure: $t_k=k\epsilon$, $\epsilon=T/M$, $k=0,\dots,M$, $\mathcal{D}\boldsymbol{\rho}_i(t)=\Pi_{k=1}^{M-1}d\boldsymbol{\rho}_i(t_k)$, $\boldsymbol{\rho}_i(t_0)=\mathbf{r}_i$, $\boldsymbol{\rho}_i(t_M)=\mathbf{R}_i$, $\mathcal{D}\mathbf{p}_i(t)=\Pi_{k=1}^{M}d\mathbf{p}_i(t_k)$, $M\to\infty$,

$$\mathcal{S}=\sum_{k=0}^{M-1}\left[\frac{\epsilon}{2}p_i^\alpha(t_{k+1})[\mathcal{K}]_{ij}^{\alpha\beta}(t_k)p_j^\beta(t_{k+1})-p_i^\alpha(t_{k+1})\times[\rho_i^\alpha(t_{k+1})-\rho_i^\alpha(t_k)]\right], \tag{2.13}$$

where the path integral for the associated quantum mechanics could be understood as explaining a random (Brownian) motion of $2n$ particles possessing a very special dependence of the tensor of inverse masses $[\hat{\mathcal{K}}]$ on displacements of all the particles. The resolvent represents the probability for the $2n$ fluid particles to diffuse from the initial geometry $\mathbf{r}_i$ to the final one $\mathbf{R}_i$ for time T. Notice that another $2n$-particle representation [20] was used to analyze the pumping-free (decaying turbulence) two-dimensional case of a linear ($\gamma=0$) anisotropic velocity profile.

The representation (2.10)–(2.13) is useless if we aim to calculate the functional integral explicitly: it would reduce one to calculation of the resolvent of $\hat{\mathcal{L}}$, which is already stated as a generally unsolved problem. Our aim is modest, we are going to study the higher correlation functions, or many-particle problem ($n\gg d$) in the language of a ''quasiclassical'' approximation for the associated quantum mechanics. The large parameter should allow us to evaluate the path integral from the integrand of Eq. (2.10) (or its spatial derivatives, see below) in a saddle-point (instanton) manner.

F_{2n} is not scale invariant. The integrations over $\mathbf{R}$ and T in Eq. (2.8) give rise to a huge set of zero modes, describing not only the $2n$th structural function but all the lowest ones too (for details of the zero-mode ideology see [4–7]). To separate zero mode giving the dominant contribution into the $2n$th structural function, which is subleading in the zoo of the zero modes, we suggest another oblique way of solving the problem. The idea is to use an exact scaling relation between the nth order moment of the dissipation field $\varepsilon=\kappa[\boldsymbol{\nabla}\theta]^2$ and $2n$th order structural function of the scalar that was proved in [6] by means of the ultraviolet fusion rules discovered in [4]:

$$\text{if } \langle(\theta_1-\theta_2)^{2n}\rangle \sim r_{12}^{n\gamma}(L/r_{12})^{\Delta_{2n}}, \quad \text{then } \langle\varepsilon^n\rangle \sim (L/r_d)^{\Delta_{2n}}, \tag{2.14}$$

if it is known additionally that $\langle\varepsilon^n\rangle$ is scale invariant. Let us emphasize that the relation (2.14) between structural function and respective correlation function of ε is based crucially on the expected scale invariance of both the objects. At n, considered to be the largest number in the theory, the scale invariance over r/L does not need to be valid *a priori*.

We will construct an instanton for the correlator of the dissipation field itself,

$$\langle\varepsilon^n\rangle \sim \kappa^n \lim_{r_i\to 0}\left[\int_0^\infty dT \int \prod_i d\mathbf{R}_i \mathcal{R}_\varepsilon(T;\mathbf{r}_i,\mathbf{R}_i)\chi_{2n}(\mathbf{R}_i)\right], \tag{2.15}$$

$$\mathcal{R}_\varepsilon\{T;\mathbf{r}_i,\mathbf{R}_i\} = \int_{\boldsymbol{\rho}_i(0)=\mathbf{r}_i}^{\boldsymbol{\rho}_i(T)=\mathbf{R}_i} \prod_i^{2n} \mathcal{D}\boldsymbol{\rho}_i(t)\mathcal{D}\mathbf{p}_i(t) \times \prod_{k=1}^{n}[\mathbf{p}_{2k-1}(0)\mathbf{p}_{2k}(0)] \times \exp[-\mathcal{S}(\boldsymbol{\rho}(t);\mathbf{p}(t))]. \tag{2.16}$$

It is easy to check by means of direct Gaussian integrations that the discretization condition (2.13) reproduces the correct gradient structure of the ε correlation function (the Hamiltonian form of the path integral allows it to be easily checked). A kind of pairing of the space indexes in the $p-p$ integrand of Eq. (2.16) is arbitrary (for example, one could make the integrand symmetric with respect to all permutations of all the particles).

There are two different specifications that we are free to fix in the problem's set. It concerns initial and final conditions imposed. The initial condition is defined by the initial $\mathbf{r}_i$ geometry. The final condition is defined by the source function χ_{2n}. However, scaling exponents do not depend on a concrete form of the χ function in accordance with the general zero-mode ideology [4–7]. One can use the freedom to make an appropriate choice of the initial geometry and the source function.

It is evident that integration over $\mathbf{R}_i$ in Eq. (2.15) cannot be performed in the saddle-point manner, if the source function is, for example, a uniform constant inside the circle $R<L$: All the values of $\mathbf{R}$ satisfied, $R^\gamma \lesssim DT$ (the rough observation will be improved later on), give comparable weights in the integrand of Eq. (2.15). However, one can force a particular final geometry $\mathbf{R}_i\sim L$ to be preferable, choosing the source function to get a sharp maximum about $R\sim L$, where R is an average size, say $R=\sqrt{(\Sigma_i \mathbf{R}_i^2)/(2n)}$. Then one can include the variation over $\mathbf{R}_i$ in the common variation procedure adding the term $-\ln(\chi_{2n})$ to the action. The formal trick is justified by the general expectation to get the dominant contribution into the ε correlation function from a zero mode of operator $\hat{\mathcal{L}}$. It is the universal scaling of a zero mode that defines universal (independent on a concrete shape of χ_{2n}) scaling of the ε correlation function.

Thus we are going to raise both the $\mathbf{p}-\mathbf{p}$ and source terms from the integrand of Eq. (2.16) into the exponent to vary hereafter the effective action

$$\mathcal{S}_{\rm eff} = \mathcal{S} - \sum_{k=1}^{n} \ln(\kappa \mathbf{p}_{2k-2}\mathbf{p}_{2k-1}) - \ln(\chi_{2n}), \tag{2.17}$$

over all the allowed trajectories [over $\boldsymbol{\rho}_i(t),\mathbf{p}_i(t)$ for all the t from $0\le t\le T$] in the next section.

III. SADDLE-POINT APPROXIMATION

An instanton is defined by extremum of the effective action (2.17) with respect to fluctuating coordinates $\boldsymbol{\rho}_i(t)$ and momenta $\mathbf{p}_i(t)$ of all the $2n$ particles:

$$\dot{p}_i^\alpha + p_i^\beta p_j^\eta \mathcal{K}_{ij}^{\beta\eta;\alpha} = \delta(T-t)\frac{\partial \ln(\chi_{2n})}{\partial \rho_i^\alpha(T)}, \tag{3.1}$$

$$\dot{\rho}_i^\alpha - [\mathcal{K}]_{ij}^{\alpha\beta} p_j^\beta = -\frac{p_{i*}^\alpha}{\mathbf{p}_j\mathbf{p}_{j*}}\delta(t), \tag{3.2}$$

where summations over the particle j index and repeated spatial indexes are supposed; j and $j*$ are indices of conjugated particles from a pair (say 1 and 2 or $2n-1$ and $2n$);

$$\mathcal{K}^{\beta\eta;\alpha}(\boldsymbol{\rho}) \equiv \frac{\partial}{\partial\rho^\alpha}\mathcal{K}^{\beta\eta}(\boldsymbol{\rho}) = \frac{D}{\rho^\gamma}\left((d+1-\gamma)\rho^\alpha\delta^{\beta\eta} - \delta^{\alpha\beta}\rho^\eta - \delta^{\alpha\eta}\rho^\beta + \gamma\frac{\rho^\alpha\rho^\beta\rho^\eta}{\rho^2}\right). \tag{3.3}$$

The discrete variant of the instanton equations is

$$\epsilon[\mathcal{K}]_{ij}^{\alpha\beta}(t_k)p_j^\beta(t_{k+1}) + \rho_i^\alpha(t_k) - \rho_i^\alpha(t_{k+1}) = 0, \tag{3.4}$$

$$\epsilon p_i^\eta(t_{k+1})\mathcal{K}_{ij}^{\eta\beta;\alpha}(t_k)p_j^\beta(t_{k+1}) + p_i^\alpha(t_{k+1}) - p_i^\alpha(t_k) = 0, \tag{3.5}$$

$$r_i^\alpha + \frac{p_{i*}^\alpha(\epsilon)}{\mathbf{p}_i(\epsilon)\mathbf{p}_{i*}(\epsilon)} = \rho_i^\alpha(t_1), \tag{3.6}$$

$$p_i^\alpha(t_M) = -\frac{\partial \ln[\chi_{2n}]}{\partial\rho_i^\alpha(t_M)}, \tag{3.7}$$

where k, the temporal index in Eqs. (3.4) and (3.5), is running from 1 to $M-1$; $t_1=\epsilon=0^+$. Equations (3.6) and (3.7), appearing from $\partial\mathcal{S}_{\rm eff}/\partial p_i^\alpha(t_1)=0$, and $\partial\mathcal{S}_{\rm eff}/\partial\rho_i^\alpha(t_M)=0$, respectively, explain the rule of parametrization of the δ functions from the right hand sides of Eqs. (3.2) and (3.1). To study the saddle-point trajectories at the fixed initial geometry $\mathbf{r}_i$ and a fixed form of the source function χ_{2n} one should solve the following classical equations of motion:

$$\dot{p}_i^\alpha + p_i^\beta p_j^\eta \mathcal{K}_{ij}^{\beta\eta;\alpha} = 0, \tag{3.8}$$

$$\dot{\rho}_i^\alpha = [\mathcal{K}]_{ij}^{\alpha\beta} p_j^\beta, \tag{3.9}$$

in the boundary conditions

$$\boldsymbol{\rho}_i(0) = \mathbf{r}_i', \quad \boldsymbol{\rho}_i(T) = \mathbf{R}_i, \tag{3.10}$$

where $\mathbf{r}_i' \equiv \boldsymbol{\rho}_i(\epsilon)$ is related to $\mathbf{r}_i$ and the initial momentum $\mathbf{p}_i(\epsilon \to 0)$ via Eq. (3.6); $\mathbf{R}_i \equiv \boldsymbol{\rho}_i(t_M)$ depends on the final momentum $\mathbf{p}_i(t_M)$ via Eq. (3.7). Therefore the problem is reduced to resolving Eqs. (3.8) and (3.9) with the boundary condition (3.10) fixed and with the constraints (3.6) and (3.7) imposed afterwards.

The classical Hamiltonian equations of motion (3.8) and (3.9) possess the standard set of integrals of motion. First of all, due to the independence of the Lagrangian (the integrand part of the action $\mathcal{S}$) on time, the energy (that coincides with the Lagrangian) is the conserved quantity

$$E = -\frac{1}{2}\dot{\rho}_i^{\alpha} p_i^{\alpha} = \text{const.} \tag{3.11}$$

Second, due to the invariance of the action with respect to a uniform shift of all the particles, the momentum $P^{\alpha} = \Sigma_i p_i^{\alpha}$ is conserved too. Third, due to the invariance of the action with respect to uniform rotation, the angular momentum

$$M^{\alpha_3, \ldots, \alpha_d} = \rho_i^{\alpha_1} \epsilon^{\alpha_1, \ldots, \alpha_d} p_i^{\alpha_2} = \text{const}, \tag{3.12}$$

which is a $(d-2)$-dimensional antisymmetric tensor ($\epsilon^{\alpha_1, \ldots, \alpha_d}$ is the d-dimensional absolutely antisymmetric tensor), is the last globally conserved quantity.

A. Semiclassical analysis for correlation functions

Let us consider a particular instanton solution describing dynamical dispersion of particles from a geometry with $\rho_{ij} \sim r'$ at the initial moment of time 0^+, to a final geometry of a common type; with at least one of the distances $\rho_{ij}(T)$ being of the order of $R \sim L$. The major saddle-point contribution into the ε correlator is

$$\langle \varepsilon^n \rangle^{\text{cl}} \sim \kappa^n \lim_{r \to 0} \left[\int_0^{\infty} dT \, \exp(-S_{\text{eff}}^{\text{cl}}) \right], \tag{3.13}$$

where $\mathcal{S}_{\text{eff}}^{\text{cl}}$ is the effective action $\mathcal{S}_{\text{eff}}$ (2.17) taken on the classical trajectory $\boldsymbol{\rho}_i^{\text{cl}}$. The law of temporal evolution of any such trajectory is

$$\rho^{\gamma/2} - r'^{\,\gamma/2} \sim R^{\gamma/2} \frac{t}{T}, \tag{3.14}$$

$$p \sim \frac{R^{\gamma/2}}{nDT\rho^{1-\gamma/2}}, \tag{3.15}$$

$$\mathcal{S}^{\text{cl}} \sim \frac{nR^{\gamma}}{\alpha(n)TD}, \tag{3.16}$$

where $0^+ < t < T$, and $\alpha(n)$ will be defined in the next paragraph. The proportionality signs $\sim$ in Eqs. (3.14)–(3.16) stand to point out that Eqs. (3.14)–(3.16) are correct up to constant multipliers, depending on the details of the geometry. Still, there are no extra scale and n dependences in the explicit version of Eqs. (3.14)–(3.16). The universal (with respect to the geometry's variation) scaling behavior, $\rho \sim t^{2/\gamma}$, follows from equations (any one of) (3.8) and (3.9), by substituting there scaling (over time) ansatz for both ρ and p fields supported by the energy conservation law (3.11). Here in Eqs. (3.14)–(3.16), considering all the distances to be much larger than r_d we dropped diffusion for a while. One accounts for diffusion in a special symmetrical case (Appendix A 1) aiming to show that to get the principal dependence of an ultraviolet divergent quantity on r_d it is enough, generally, just to replace all the separations going to 0 by r_d (see also [4]).

We consider two very symmetrical cases in Appendix A: (1) the uniform expansion of an S_m sphere ($2n$ points uniformly distributed on the sphere), the $m \leq d$; (2) divergence of two drops, with n_+ and $2n - n_+$ particles merged in the first and second points (drops), respectively. One can separate all the possible trajectories into two different types dependent on how $\alpha(n)$ behaves with n going to ∞. Most of the trajectories, we will call them "typical," correspond to a linear growth of $\alpha(n)$ with n. To specify, the trajectory is "typical" if the volume bounded by a smooth $(d-1)$-dimensional manifold built on the $2n$ points is not temporarily increased. Relative divergence of the two-point geometry (see Appendix A 3), the same as expansion of the S_m geometry with $m < d-1$, are typical. An example of "nontypical" trajectory is expansion of the S_d sphere (see Appendixes A 1, A 2 for an explanation of S_m geometry). $\alpha(n)/n$ decreases with n going to ∞ for a nontypical trajectory.

For a specific kind of source function χ (possessing a sharp maximum) chosen, R appears to be $\sim L$. There are two different intervals over the integral time T. First, $r' = \rho(0^+)$ governed by Eq. (3.6) is about r at $0 < T \ll L^{\gamma/2} r^{\gamma/2}/[D\alpha(n)]$; For the largest values of the integral time, $L^{\gamma/2} r^{\gamma/2}/[D\alpha(n)] \ll T$, one gets $p(0) = 1/r$ and $r' \sim \{rL^{\gamma/2}/[DT\alpha(n)]\}^{2/(2-\gamma)}$. By substituting the saddle-point values, governed by Eqs. (3.14)–(3.16), into Eq. (3.13) one gets a divergence in the integral at the largest times. The divergence is formal: it should be stabilized by the normalization factor, accounting for an algebraic decay of the resolvent with T. We will see below (in Sec. IV) that the algebraic factor in fact comes into the game via accounting for fluctuations to cut the temporal integration in Eq. (3.13) at $T \sim L^{\gamma}/[\alpha(n)D]$. Thus the classical action scales linear with n. Thus for all the values of γ except some vicinities of the "diffusive" $\gamma = 2$ and "logarithmic" $\gamma = 0$ limits one gets

$$\langle \varepsilon^n \rangle^{\text{cl}} \sim (L/r_d)^{n\gamma}. \tag{3.17}$$

Equation (3.17) accounts for the principal dependence of $\langle \varepsilon^n \rangle^{\text{cl}}$ on the Péclet number only. It is the second interval (of the largest values) of T that gives the dominant contribution in Eq. (3.17): All the significant dependence on r [or on r_d after taking the limit on the right-hand side of Eq. (3.13)] in the integrand of Eq. (3.13) comes from the p_0^{2n} term via the multiplier r^{2n}. Finally, accounting for the dependence of κ on the diffusion scale results in the anomalous result (3.17). Note that the saddle-point result (3.17) is generic for all the space dimensions $d \geq 2$.

What is specific about some vicinity of $\gamma_d = 2^-$ is an expected inapplicability of the saddle-point approximation there: A growth of $\mathcal{S}_{\text{eff}}^{\text{cl}}$ with a growth in n is diminished by decay of the action as γ goes to 2^-. Note that in the naive diffusion limit, $D = 0$ [or in the special limit $\gamma = 2^-$, $D/(2-\gamma) = \text{const}$, see [5]] a balance between different terms in Eq. (3.6) differs strongly from the general case: there the r term can be dropped in comparison with r-independent ones. r' turns out to be of the order of the pumping scale L, that results in the absence of anomalous scaling (as it should be: the diffusion case is Gaussian). An infinitesimally small deviation of γ from 2^- in the special limit of [5] makes the $p_0 \sim 1/r$ anomalous solution preferable in comparison with the nonanomalous $p_0 \sim 1/L$ one. Thus the $\gamma \to 2^-$ limit is indeed very peculiar. The logarithmic $\gamma=0$ limit is very specific too. What is written above in the general scheme is valid if $[(r'/L)^{\gamma/2} - (r'/L)^{\gamma}]/\gamma \ll 1$ is satisfied. However, the inequality ceases to be true at some very tight vicinity of $\gamma=0$: The second term from the left-hand side of Eq. (3.6) could be dropped there [the condition is opposite to the one which resulted in Eq. (3.17)].

The anomalous answer (3.17) is generic: the scale invariance holds true for the classical trajectories of both typical and nontypical kinds. It accounts for the n-dependent prefactor in the integrand of Eq. (3.13), discriminating between the two kinds of instantons. At R and T considered to be fixed, the r-independent term of $\mathcal{S}_{\text{eff}}^{\text{cl}}$ gets no n dependence in the first case of the typical instanton (see Appendixes A 2, A 3 for the two-point instanton and Appendix A 1 for the S_m spherical case with $m>d$). Vice versa, the nontypical instanton (it is a spherical case with $m=d$, for example) gets n dependence from the bare action (3.16). $\alpha(n)$ goes to zero as n goes to ∞ in this case. To conclude, the nontypical instanton (S_d one) is suppressed in comparison with the typical ones. However, we cannot distinguish between different typical instantons (the S_m instanton with $m>d$ and the two-point instanton) on the classical level. The suppression of a nontypical instanton in d dimensions has a clear physical explanation. It follows from the conservation of the volume of a fluid element, prescribed by the incompressibility condition. It is a very rare trajectory (that means it has a small weight) that stretches the $2n$ points forming the S_d sphere and conserves simultaneously the volume enveloped by a $d-1$ surface built on the $2n$ points. The surface cannot be smooth in this case, it is very fractal.

Note that at $1<\gamma<2$ and $d=2$ there is not another symmetrical instanton of the type discussed above except the S_2 one. The two-point instanton that works pretty well at $0<\gamma<1$ turns out to be unstable at $\gamma>1$: particles being initially dropped together into a group try to diverge from each other hereafter. Most probably it is reasonable to study another symmetrical instanton with all the particles being elongated into a straight line in this case. Such an instanton could be preferable in comparison with the S_2 one. We do not yet consider the straight-line instanton in the present paper, postponing it for a future study.

For any solution (already discussed or another) of the auxiliary problems (3.8)–(3.10) one can design such an appropriate initial geometry $\mathbf{r}_i$ and a particular form of the source function χ_{2n} that the trajectory turns out to be a unique solution of the full system of the saddle-point equations (3.1) and (3.2): Fixing $\mathbf{r}_i'$ and $\mathbf{p}_i(0)$ one arrives at a unique [due to constraint (3.6)] initial geometry. Via explicit dynamics and the second constraint (3.7), one finds an appropriate form of the source function to make the saddle-point solution self-consistent.

The anomalous answer (3.17) is scale invariant. In the leading "classical" approximation the anomalous scaling is extraordinarily large: The normal contribution to ζ_{2n} is fully compensated by the anomalous one, $\Delta_{2n}^{\text{cl}} \to n\gamma$ at $n\to\infty$. However, the result gives no possibility for answering the major questions: Is the ε correlator scale invariant at $n\to\infty$? And if it is scale invariant, what is the asymptotic of ζ_{2n} at the largest n? Indeed, it is obvious to expect that the small parameter of the theory (the parameter that allows one to turn down non-Gaussian fluctuations on the ground of Gaussian ones) is $\sim 1/\sqrt{n}$. However, it is not obvious to expect that the expansion of the anomalous exponent about the saddle-point value $\Delta_{2n}^{\text{cl}} = n\gamma$ is a series in $1/\sqrt{n}$. The only way to resolve this problem is to account for fluctuations directly, following dependence of the fluctuation factor on the Péclet number.

It was established in the present section that the variety of saddle-point solutions (for different $\mathbf{r}_i$ and χ) gives the same scale dependence (3.17). It supports the general statement [4,6] that the dominant contribution to $\langle \varepsilon^n \rangle$ stems from a scale-invariant zero mode of $\hat{\mathcal{L}}_{2n}$. However, it follows from the same general statement too, that the dominant contribution can be lacking for a special kind of source function and initial geometry. It is along this pathways that one should optimize the problem with respect to $\mathbf{r}_i$ and the χ function, to find the dominant zero-mode contribution: We must not only find a contribution of fluctuations in $\langle \varepsilon^n \rangle$ (which should be small with respect to n in comparison with the saddle-point solution) but show it is maximal with respect to Pe.

IV. ACCOUNTING FOR FLUCTUATIONS

Let us study in the path integral (2.16) Gaussian fluctuations about an as yet unspecified classical trajectory ($\boldsymbol{\rho}^{\text{cl}}$ and $\mathbf{p}^{\text{cl}}$ are supposed to be known). The quadratic with respect to fluctuated fields $\delta\boldsymbol{\rho}$,$\delta\mathbf{p}$, a correction to the classical action $\mathcal{S}_{\text{eff}}^{\text{cl}}$, is

$$\delta\mathcal{S}_{\text{eff}} = \frac{1}{2}\int_0^T dt\Big(\delta p_i^\alpha \mathcal{K}_{ij}^{\alpha\beta}\{\boldsymbol{\rho}^{\text{cl}}\}\delta p_j^\beta + 2\,\delta p_i^\alpha \mathcal{A}_{ij}^{\alpha\beta}\delta\rho_j^\beta + \delta\rho_i^\alpha \mathcal{B}_{ij}^{\alpha\beta}\delta\rho_j^\beta - 2\,\delta p_i^\alpha \delta\dot{\rho}_i^\alpha\Big) - \frac{1}{2}\,\delta p_i^\alpha(0)\mathcal{G}_{ij}^{\alpha\beta}\delta p_j^\beta(0) + \frac{1}{2}\,\delta\rho_i^\alpha(T)\mathcal{C}_{ij}^{\alpha\beta}\delta\rho_j^\beta(T), \tag{4.1}$$

$$\mathcal{A}_{ij}^{\alpha\beta} = \delta_{ij}\sum_k \mathcal{K}_{ik}^{\alpha\nu;\beta}\{\boldsymbol{\rho}^{\text{cl}}\}p_k^{\text{cl}^\nu} - \mathcal{K}_{ij}^{\alpha\nu;\beta}\{\boldsymbol{\rho}^{\text{cl}}\}p_j^{\text{cl}^\nu},$$

$$\mathcal{B}_{ij}^{\alpha\beta} = \delta_{ij}\sum_k p_i^{\text{cl}^\nu}\mathcal{K}_{ik}^{\nu\mu;\alpha\beta}\{\boldsymbol{\rho}^{\text{cl}}\}p_k^{\text{cl}^\mu} - p_i^{\text{cl}^\nu}\mathcal{K}_{ij}^{\nu\mu;\alpha\beta}\{\boldsymbol{\rho}^{\text{cl}}\}p_j^{\text{cl}^\mu}, \tag{4.2}$$

$$\mathcal{C}_{ij}^{\alpha\beta} = -\left.\frac{\partial^2 \ln(\chi_{2n})}{\partial\rho_i^\alpha \partial\rho_j^\beta}\right|_{\boldsymbol{\rho}^{\text{cl}}(T)}, \qquad \mathcal{G}_{ij}^{\alpha\beta} = -\frac{\delta^{\alpha\beta}\delta_{ji*}}{\mathbf{p}_i^{\text{cl}}(0)\mathbf{p}_{i*}^{\text{cl}}(0)}, \tag{4.3}$$

where there is no summation over repeated j indices in Eq. (4.2); the pair of the particles' indices i and $i*$ describe a conjugated pair of particles; $\mathcal{K}_{\mathbf{r}}^{\alpha\beta;\nu}$ and $\mathcal{K}_{\mathbf{r}}^{\alpha\beta;\nu\mu}$ stand for the first and second spatial derivatives of $\mathcal{K}_{\mathbf{r}}^{\alpha\beta}$ [see explicit expression for the first derivative (3.3)]. Performing the Gaussian integrations over $\delta\boldsymbol{\rho},\delta\mathbf{p}$ one arrives at the following expression for the ε correlator accounting for the Gaussian fluctuations:

$$\langle\varepsilon^n\rangle\sim\lim_{r\to r_d}\left[\int_0^\infty dT\,\exp(-\mathcal{S}_{\rm eff}^{\rm cl})\mathcal{Z}^{\rm fl}\right],\quad \mathcal{Z}^{\rm fl}=\langle\Psi_0|\Psi_T\rangle, \tag{4.4}$$

$$\langle\Psi_0|=\left\langle\prod_i\ \delta(\widetilde{\mathbf{r}}_i)\right|\exp\left[\frac{1}{2}\,\mathcal{G}_{ij}^{\alpha\beta}\boldsymbol{\nabla}_{\mathbf{r}_i}^{\alpha}\boldsymbol{\nabla}_{\mathbf{r}_j}^{\beta}\right], \tag{4.5}$$

$$|\Psi_T\rangle=\widetilde{T}\exp\left[\int_0^T\delta\hat{\mathcal{L}}dt\right]|\delta\chi(\widetilde{\mathbf{r}})\rangle,$$

$$\delta\chi(\widetilde{\mathbf{r}})=\exp\left[-\frac{1}{2}\,\widetilde{\mathbf{r}}_i^{\alpha}\mathcal{C}_{ij}^{\alpha\beta}\widetilde{\mathbf{r}}_j^{\beta}\right], \tag{4.6}$$

$$\delta\hat{\mathcal{L}}\{t;\widetilde{\mathbf{r}}\}=-\frac{1}{2}\sum_{i,j}\ (\mathcal{K}_{ij}^{\alpha\beta}\{\boldsymbol{\rho}^{\rm cl}\}\boldsymbol{\nabla}_{\widetilde{\mathbf{r}}_i}^{\alpha}\boldsymbol{\nabla}_{\widetilde{\mathbf{r}}_j}^{\beta}+2\mathcal{A}_{ji}^{\beta\alpha}\widetilde{r}_i^{\alpha}\boldsymbol{\nabla}_{\widetilde{\mathbf{r}}_j}^{\beta}$$
$$-\widetilde{r}_i^{\alpha}[\mathcal{B}]_{ij}^{\alpha\beta}\widetilde{r}_j^{\beta}), \tag{4.7}$$

$$[\hat{\mathcal{B}}]=\hat{\mathcal{B}}-\hat{\mathcal{A}}^T\hat{\mathcal{K}}^{-1}\hat{\mathcal{A}}. \tag{4.8}$$

Here in Eq. (4.4), we use the canonical quantum mechanical notations for matrix elements. The operator in Eq. (4.5) is the descendant of the momentum's term from the integrand of Eq. (2.16); the diffusivelike state (4.5) is well defined. $\widetilde{T}$exp in Eq. (4.6) stands for an antichronological ordered exponential. Thus we came full circle at this stage of the calculations, returning back to a problem in the operator representation form [compare the time-ordered exponential from Eq. (4.6) with the original operator exponent, say from the first line of Eq. (2.8)]. It follows from Eq. (4.6) that Ψ_T can be understood as a solution for the differential equation

$$(\partial_t+\delta\hat{\mathcal{L}})\Psi(t;\widetilde{\mathbf{r}})=\delta(t-T)\,\delta\chi(\widetilde{\mathbf{r}}), \tag{4.9}$$

at an initial moment of time $\Psi_T(\widetilde{\mathbf{r}})=\Psi(0;\widetilde{\mathbf{r}})$.

Let us consider fluctuations about a typical saddle-point trajectory with all the distances stretched somehow similarly (we will specify the concrete form of the considered instantons later on). Performing rescaling of temporal and spatial variables in Eq. (4.9) one simplifies it. In the new dimensionless $\tau,\mathbf{s}_i$ variables

$$\tau=\frac{R^{\gamma/2}}{T}\int_0^t\frac{dt}{\rho^{\gamma/2}}=\frac{\gamma}{2}\ln(\rho/r'),\quad \mathbf{s}_i=\frac{R^{\gamma/4}\widetilde{\mathbf{r}}_i}{\sqrt{DT}\rho^{1-\gamma/4}}, \tag{4.10}$$

where $\rho(t)$ is a typically stretched, Eq. (3.14), classical trajectory, Eq. (4.9) gets the refined form

$$(\partial_\tau+\hat{\mathcal{L}}')\Psi(\tau;\mathbf{s}_i)=\delta\!\left(\tau-\frac{\gamma}{2}\ln(R/r')\right)\delta\chi(\mathbf{s}_iR^{1-\gamma/2}\sqrt{TD}),$$
$$0\leqslant\tau\leqslant\ln(R/r'), \tag{4.11}$$

$$\Psi_T=\Psi(0;\mathbf{s}_i=\sqrt{D/T}\,R^{\gamma/4}\widetilde{\mathbf{r}}_i/r'^{\,1-\gamma/4}), \tag{4.12}$$

$$\hat{\mathcal{L}}'=\frac{1}{2}\{-\widetilde{\mathcal{K}}_{ij}^{\alpha\beta}\boldsymbol{\nabla}_{\mathbf{s}_i}^{\alpha}\boldsymbol{\nabla}_{\mathbf{s}_j}^{\beta}+2\widetilde{\mathcal{A}}_{ji}^{\beta\alpha}s_i^{\alpha}\boldsymbol{\nabla}_{\mathbf{s}_j}^{\beta}-s_i^{\alpha}\widetilde{\mathcal{B}}_{ij}^{\alpha\beta}s_j^{\beta}\}$$
$$-\frac{4-\gamma}{2\gamma}\,s_i^{\alpha}\boldsymbol{\nabla}_i^{\alpha}, \tag{4.13}$$

$$\widetilde{\mathcal{K}}_{ij}^{\alpha\beta}=\frac{1}{\rho^{2-\gamma}}\mathcal{K}_{ij}^{\alpha\beta}\{\boldsymbol{\rho}^{\rm cl}\},\quad \widetilde{\mathcal{A}}_{ji}^{\beta\alpha}=\frac{DT\rho^{\gamma/2}}{R^{\gamma/2}}\mathcal{A}_{ji}^{\beta\alpha},$$
$$\widetilde{\mathcal{B}}_{ij}^{\alpha\beta}=\frac{D^2T^2\rho^2}{R^{\gamma}}[\mathcal{B}]_{ij}^{\alpha\beta}, \tag{4.14}$$

where all the new dimensionless matrixes $\widetilde{\mathcal{K}},\widetilde{\mathcal{A}},\widetilde{\mathcal{B}}$ are time (τ) independent. If we exclude divergent degrees of freedom (we should worry about uniform rotation of the classical trajectory, that is a soft mode, separately) from $\hat{\mathcal{L}}'$, it becomes a Hamiltonian of a well posed quantum mechanics. It is the quantum mechanics of l Gaussian oscillators, $l\lesssim 2nd$. There is thus seen to be a gap in the spectrum of the reduced operator. There are two essential (for present consideration) characteristics of the energy spectrum: the value of the gap Δ_E and the level spacing δ_E between the ground state and the lowest excited state. Both the energetic characteristics are positive functions of n,d,γ. Stretching time $\tau=\gamma\ln(L/r')/2$ is a big parameter due to Pe$\gg d$.

There exist two different situations depending on how δ_E behaves at the largest n. First, $\tau\delta_E$ is a large parameter if δ_E does not decay as n grows. Then, it is the evolution of the ground state $\Psi_{\rm gr}\{\mathbf{s}_i\}$ giving the major contribution to Ψ_T,

$$\Psi_T=\Psi_{\rm gr}\{\sqrt{DT}\,R^{\gamma/4}\widetilde{\mathbf{r}}_i/r'^{\,1-\gamma/4}\}$$
$$\times\left(\frac{r'}{R}\right)^{\gamma\Delta_E/2}\langle\Psi_{\rm gr}\{\mathbf{s}_i\}|\,\delta\chi(\mathbf{s}_iR^{1-\gamma/2}\sqrt{TD})\rangle. \tag{4.15}$$

The multiplier $\mathcal{Z}^{\rm fl}$ is getting smaller with r algebraically,

$$\mathcal{Z}_{c-t}^{\rm fl}=\int\prod_i\ d\widetilde{\mathbf{r}}_i\ \frac{\exp(-\widetilde{r}_i^{\alpha}[\hat{\mathcal{G}}^{-1}]_{ij}^{\alpha\beta}\widetilde{r}_j^{\beta}/2)}{\sqrt{|\det[\hat{\mathcal{G}}]|}}\ \Psi_T^{(1)}$$
$$\sim\left[\frac{r'}{L}\right]^{\gamma\Delta_E/2}\left[\frac{r'^{\,1-\gamma/4}\sqrt{TD}}{rL^{\gamma/4}}\right]^l$$
$$\sim\left(\frac{r}{L}\right)^{(l/2+\Delta_E)\gamma/(2-\gamma)}\left[\frac{n^{2-\gamma/2}DT}{L^{\gamma}}\right]^{-l/(2-\gamma)}, \tag{4.16}$$

where a typical matrix element of $[\hat{\mathcal{G}}^{-1}]_{ij}^{\alpha\beta}$ is estimated as $[p^{\rm cl}(t=0)]^2\sim r^{-2}$, T is considered to be smaller than L^{γ}/D, and l counts the number of the stretched degrees of freedom. The second possibility is realized if δ_E is getting smaller with n going to ∞. Hence it follows that one gets an evolution of a mixed wave packet built from some amount of the lowest eigenstates: The wave function of the ground state $\Psi_{\rm gr}$ in Eq. (4.15) must be replaced by the wave function of the packet. However, the multiplier $\mathcal{Z}^{\rm fl}$ is algebraic again and the characteristic size of the packet has the same parametric dependence as before. The parametric estimation (4.16) thus remains intact. The number of the stretched degrees of freedom and the value of the gap need to be specified in Eq. (4.16) to describe the anomalous exponent quantitatively.

We start the quantitative analysis from discussion of the two-point instanton that is realized at $1>\gamma>0$ only. It is an example of an instanton of the first type with both δ_E and Δ_E being of the order of unity (with respect to n). The dynamics of the two-point instanton is characterized by the relative divergence of the points (drops) along with a simultaneous contraction of the sizes of the drops (see Sec. III A and Appendix A 3). This means there are three different types of fluctuation degrees of freedom in this case. First of all they are longitudinal fluctuations of the stretched degree of freedom; second, fluctuations of the points (whole drops) in the $d-1$ directions transversal to the stretched one; and, third, fluctuations of all the rest $(2n-1)d$ contracted degrees of freedom (intrinsic fluctuations of the drops). One can calculate relative fluctuations of the drops and inner fluctuations of the drops themselves independently (it is easy to check afterwards that nondiagonal terms are negligible). Gaussian integrations account for relative longitudinal fluctuations of the point forms

$$\mathcal{Z}_{str}^{\mathrm{fl}}\sim(r/L)^{(1/2+\Delta_E^{st})\gamma/(2-\gamma)}\left[\frac{n^{2-\gamma/2}DT}{L^{\gamma}}\right]^{-1/(2-\gamma)}, \tag{4.17}$$

with Δ_E^{st} as calculated in Appendix B 1. One finds that the $d-1$ transversal fluctuations cannot be considered as Gaussian ones (attempts to restrict their study by a Gaussian level leads to divergence, see Appendix B 1). Hopefully, one can calculate the transversal non-Gaussian fluctuations explicitly. First, accounting for the relative fluctuations of the points (drops), at the initial $(t=0^+)$ and final geometries fixed, is performed by the method described in Appendix C. Second, one can account for the transversal fluctuations of $\mathbf{R}_i$ and $\mathbf{r}_i'$ explicitly too, calculating a variety of rotating classical trajectories with T, R, and r' taken from the rotationless trajectory, whereas the angular momentum (3.12) is nonzero [the trajectories are found from the auxiliary classical problem, Eqs. (3.8) and (3.9), but not from the full one, Eqs. (3.1) and (3.2), with a fixed form of the source function corresponding to the rotationless configuration]. As a result, the trajectories with nonzero angular momentums give the same value of the classical action in the leading order in $(r'/L)^{\gamma}$, as for the directly stretched rotationless case (see explicit calculation for $d=2$ in Appendix A). In brief, accounting for the contribution of strongly non-Gaussian transversal fluctuations, results in the r-independent multiplier (volume of the angular group). We discuss fluctuations of the drops themselves [the rest $(2n-1)d$ fluctuating degrees of freedom] in Appendix B 2. The fluctuations are short correlated, which results in r independence of the respective contribution $\mathcal{Z}_{\delta}^{\mathrm{fl}}$ to $\mathcal{Z}^{\mathrm{fl}}$. However, the contribution (B7) shows an essential dependence on both T and n. Making substitution of Eqs. (4.17), (B7) ($\mathcal{Z}^{\mathrm{fl}}=\mathcal{Z}_{str}^{\mathrm{fl}}\mathcal{Z}_{\delta}^{\mathrm{fl}}$), and (3.16) into Eq. (4.4), and performing the integration over T in the saddle-point manner, one finally gets that the characteristic value of the integral time is getting smaller, $T\sim L^{\gamma}/(nD)\ll L^{\gamma}/D$, with $n\to\infty$.

It should be stressed once again that the result derived for $0<\gamma<1$ is not a consequence of a specially chosen initial geometry and source term—it is generic. For a majority of appropriate initial geometries and source functions, there is a special optimal configuration (that still may be difficult to find) of values and orientations of the initial momenta, making one distance diverge but all the rest converge dynamically.

As it is shown in Sec. III A and Appendix A 3, at $\gamma>1$ there exist no alternatives to the common-type stretching of all the degrees of freedom: contraction of any distance (merging of any particles in a point) leads to a singularity, which is forbidden. And yet among all the generally stretched instantons it is preferable to get ones characterized by a stretching with at least one direction (dimension) kept stretching-free (contracted): This is what we call a ''typical'' instanton. Typical symmetrical instantons explained in Sec. III A and Appendix A 3 are S_m spherical instantons with $m<d$. Thus let us apply the conducted above analysis for the case. The n dependence of the potential term from the $2n$-particle quantum mechanics, Eq. (4.13), is estimated as $\widetilde{B}_{ij}\sim 1/n^2$ for all the values of the particle index j, except for ones from a small vicinity (on the sphere) of i (every momentum is proportional to $1/n$ in the dimensionless variables). For i and j being the nearest neighbors on the S_m sphere one gets $[B]_{ij}\sim n^{\gamma/(m-1)-2}$. One can drop all the terms beside the nearest neighbors if $\gamma/(m-1)>1$. Vice versa, if $\gamma/(m-1)<1$ one can replace all the matrix elements by $\sim 1/n^2$ terms. All the kinematic matrix elements $\mathcal{K}_{ij}$ are n independent (strictly speaking for i and j being close to each other the matrix elements are even getting smaller with $n\to\infty$ than a constant, $\sim n^{-(2-\gamma)/(m-1)}$). Hence at $\gamma/(m-1)<1$ the energy characteristics are estimated as $\delta_E\sim n^{-1}$, and Δ_E as a constant, respectively. In the opposite case $\gamma/(m-1)>1$ (that is realized only if $m=2$, $\gamma>1$) one gets $\delta_E\sim n^{\gamma/[2(m-1)]-1}$, $\Delta_E\sim n^{\gamma/[2(m-1)]}$. Calculation of the n dependence of the $\mathcal{A}$ term does not change the principal dependence on n of the energy characteristics. There is the extra parameter l which enters the anomalous answer and counts the number of typically stretched degrees of freedom. For the S_m instanton one gets $l=2(m-1)n$. To conclude, contribution of fluctuations about the S_m instantons, $d-1>m\geqslant 2$, is estimated as $\sim(r/L)^{[\Delta_E+(m-1)n]\gamma/(2-\gamma)}(n^{2-\gamma/2}TD/L^{\gamma})^{-2(m-1)n/(2-\gamma)}$. This means the Gaussian correction appears to be of the same order as (even larger than) the saddle-point value (3.17), rendering the S_m saddle points smoothed out by the Gaussian fluctuations. Particularly, at $\gamma>1$ the contribution of the S_m instanton (saddle point plus Gaussian fluctuations) to the $\langle\epsilon^n\rangle$ correlator is getting smaller with Pe increase. The contribution is of no interest since it is negligible in comparison with the forced term contribution (possessing the normal scaling) that was dropped in the saddle-point approach from the very beginning. One recognizes that the saddle-point calculus is not an appropriate tool for calculations of the anomalous exponent at $\gamma>1$.

It was thus shown in the present section that at $0<\gamma<1$ the contribution of Gaussian fluctuations to $\langle\varepsilon^n\rangle$ is algebraic (scale invariant) with respect to the Péclet number ($\mathrm{Pe}=L/r_d$) and it is small with respect to n in comparison with the classical value (3.17). The scaling exponent ζ_{2n} of the scalar's structural function shows a finite limit at $n\to\infty$. The exponent is calculated explicitly

$$\zeta_\infty=\frac{\gamma}{2(2-\gamma)}+\frac{d+2-\gamma}{2(2-\gamma)}\left[2-3\gamma/2+\sqrt{16-16\gamma+17\gamma^2/4}\right]\quad \text{at } 0<\gamma<1. \tag{4.18}$$

There are relative fluctuations of the points in the two-point geometry that are responsible for the answer (4.18). At $\gamma>1$ all the saddle-point solutions discussed in Sec. III are smoothed out by the Gaussian fluctuations: The instanton calculus does not work in this case.

V. CONCLUSION

It was stated in the Introduction that the idea of the saddle-point calculus is to make n the largest number in the problem. However, to establish the criterion of applicability of the saddle-point approximation at $0<\gamma<1$ explicitly one should estimate contributions of non-Gaussian fluctuations about the instanton and compare them with the already found Gaussian corrections. If following the general scheme, Eqs. (4.4)–(4.14), to keep a nonlinear (say, third order over $\tilde{r}$ term) one arrives at an extra factor $\sqrt{TD}/[R\rho(t)]^{\gamma/4}\leqslant \mathrm{Pe}^{\gamma/4}/\sqrt{n}$ behind the dimensionless $\sim\tilde{s}^3$ term in the nonlinear variant of Eq. (4.13). The factor (along with the integral time T) is getting smaller with $n\to\infty$ (the smallness makes the saddle point become instant, and the ''classical'' action become, respectively, large). The observation is generic: all the higher order corrections to the quantum mechanics describing the Gaussian fluctuations are small if

$$\frac{\mathrm{Pe}^{\gamma/4}}{\sqrt{n}}\ll 1 \tag{5.1}$$

[here in Eq. (5.1) we do not follow the precise γ and d dependences]. The object from the left-hand side of Eq. (5.1) is the small parameter, making the saddle-point calculations valid. Particularly, the method works if n is the largest number in the problem ($n\gg d,\mathrm{Pe}$).

However, it is remarkable that the anomalous scale-invariant result (4.18) has a wider criterion of applicability than the method used for its derivation. Indeed, it is the zero mode of the primary operator $\hat{\mathcal{L}}$ that makes the major contribution to the $2n$th order structural function. Choosing the scale invariance as the key for classification of zero modes, one finds that only n, d, and γ (but not Pe) can be entered into the zero modes. Thus the anomalous result (4.18) is valid even if $\mathrm{Pe}^{\gamma/2}$ is larger than n, however, n remains much larger than d. physically. It is important to note that Eq. (4.18) matches parametrically correctly the perturbative results of the $1/d$ [4,6] expansion at n, being of the order of d.

There existed two different expectations for ζ_{2n} at n sufficiently large. Decoupling of the molecular-diffusion term in the equation (that is not closed originally) for the structural function of the $2n$th order gives $\zeta_{2n}\to\sqrt{2nd\zeta_2}$ [2,3]. Another prediction is that ζ_{2n} tends to an n-independent constant determined somehow by d and ζ_2 [21,2,22]. Thus the recent work [22] is based on an extension onto the passive scalar case of the method of operator product expansion suggested recently in the context of Burgers's turbulence [23]. Our calculations contradict the first prediction and support the prediction for ζ_{2n} to approach a constant at $n\to\infty$ if $0<\zeta_2<1$. It is worth noting that an extended prediction of [22] gives us even more than an asymptotic constant behavior for the exponent at large n: ζ_{2n} was predicted to be a constant for all the numbers n having been larger than some n_0. We cannot exclude or confirm the extended prediction here. It will require accounting for corrections to the saddle-point solution of the next non-Gaussian order explicitly. Let us stress that the equivalent problem in the Burgers turbulence has also not yet been solved.

A comparison of the first-quantized formalism (''quantum'' particles) presented in the paper with a second-quantized one (''quantum'' fields), that hopefully will be developed in accordance with the general scheme [13], would be very instructive.

It was argued phenomenologically [18,24] and confirmed quantitatively at $d\gg n$ by means of direct expansion over finite velocity's correlation time [25], that the scalar's exponents ζ_{2n} are nonuniversal, they are crucially aware of the velocity's temporal characteristics. In the present paper we developed a theory for a peculiarly adopted (to the analytical study) case of the δ-correlated Gaussian velocity field. Nevertheless, the starting technical idea of the paper to replace $\boldsymbol{\nabla}_i$ in the operator representation by momentum of the ith particle in the path integral does not require any temporal or statistical restrictions on the Eulerian velocity field being imposed. Hence it follows that it would be interesting to generalize the theory of finding ζ_{2n} at $n\gg d$ for the more realistic case of finite temporal correlations and generally for arbitrary degrees of non-Gaussianity of the velocity field. It remains to be seen whether the scaling of the largest moments is nonuniversal in the limit. It is hoped that quantitative comparison of the future theory with experimental data (say, with measurements of temperature structural functions, up to order 12 [26]; for review of experiments see [27]) will be real some day.

The theory prediction does not contradict simulations for $d=2$, $\zeta_2=1/2$ reported in [3] (there, the velocity field was swept rapidly through the scalar to mimic the short-correlated feature of Kraichnan's model). The largest tenth moment measured in the simulation gives $\zeta_{10}\approx 1.6085$, which is smaller (as it should be due to the convexity inequality) than our asymptotic result $\zeta_\infty\approx 5.1$.

ACKNOWLEDGMENTS

This work is a part of the extensive program on studying anomalous scaling in turbulence undertaken together with E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev at the Weizmann Institute. We greatly benefited from very helpful numerous discussions with them. Useful remarks of

A. Kamenev, D. Khmelnitskii, S. Levit, D. Orgad, and V. Yakhot are gratefully acknowledged. This work was supported by the Clore Foundation.

APPENDIX A: SOURCE-FREE SYMMETRICAL INSTANTONS

1. S_l sphere geometry in d dimensions

Consider $2n$ points equidistantly distributed on the l-dimensional sphere: for example, in the two-dimensional case, $d=l=2$, we can use the polar representation $\boldsymbol{\rho}_k=(\rho,\varphi_k)$, $\varphi_k=(k-1)\pi/n$, $k=1,\ldots,2n$, for the points' displacements. Let us consider the following setup: (1) the angular momentum (3.12) will be equal to zero; (2) the initial ($t=0^+$) spherical geometry will be preserved dynamically. Then, all the vector objects defined for some particle i, like $\dot{\boldsymbol{\rho}}_i$ or $\mathbf{p}_i$, are parallel to $\boldsymbol{\rho}_i$. The system of equations (3.8) and (3.9) is reduced to

$$\dot{p}-a_2Dp^2\rho^{1-\gamma}=0, \tag{A1}$$

$$\dot{\rho}+a_1Dp\rho^{2-\gamma}+2\kappa p=0, \tag{A2}$$

$$\begin{aligned} a_2&=-\frac{1}{D\rho^{1-\gamma}}n_i^\alpha n_i^\beta\sum_j n_j^\eta\mathcal{K}^{\beta\eta;\alpha}(\boldsymbol{\rho}_i-\boldsymbol{\rho}_j)\\ &=\frac{1}{2}\sum_k[2\sin(\varphi_k/2)]^{2-\gamma}\\ &\quad\times[(2d-\gamma)\sin^2(\varphi_k/2)-d-1+\gamma]>0, \end{aligned} \tag{A3}$$

$$a_1=-\frac{1}{D\rho^{2-\gamma}}n_i^\alpha\sum_j n_j^\beta\mathcal{K}^{\alpha\beta}(\boldsymbol{\rho}_i-\boldsymbol{\rho}_j)=\frac{2a_2}{2-\gamma}. \tag{A4}$$

Thus we arrive at the simple equation with the following boundary conditions imposed:

$$\frac{d}{dt}\left[\frac{\dot{\rho}}{a_1D\rho^{2-\gamma}+2\kappa}\right]+\frac{a_2D\rho^{1-\gamma}\dot{\rho}^2}{(a_1D\rho^{2-\gamma}+2\kappa)^2}=0, \tag{A5}$$

$$\rho(0)=r', \quad \rho(T)=R. \tag{A6}$$

Solution of Eq. (A5) fixed by the conditions (A6) is

$$\int_{r'}^{\rho(t)}\frac{dx}{\sqrt{a_1Dx^{2-\gamma}+2\kappa}}=\frac{t}{T}\int_{r'}^{R}\frac{dx}{\sqrt{a_1Dx^{2-\gamma}+2\kappa}}, \tag{A7}$$

in accordance with the energy conservation law (3.11) [to get the answer (A7) one could replace one of the basic equations (A1) and (A2) by the conservation law (3.11)]. On the present instanton solution (A7) the action (2.11) gets the form

$$\mathcal{S}^{\rm cl}(T;r',R)=\frac{n}{2T}\left(\int_{r'}^{R}\frac{dx}{\sqrt{a_1Dx^{2-\gamma}+2\kappa}}\right)^2. \tag{A8}$$

The momentum of the $k=0$ particle at zero moment of time is

$$p|_{t=0}=-\frac{\int_{r'}^{R}dx(a_1Dx^{2-\gamma}+2\kappa)^{-1/2}}{T\sqrt{a_1Dr'^{2-\gamma}+2\kappa}}. \tag{A9}$$

Calculating a_1 in the two-dimensional case of S_2 geometry, one finds that $a_1=[(4-\gamma)b_2/4-(3-\gamma)b_1]/(2-\gamma)$ and

$$b_1=\sum_k[2\sin(\varphi_k/2)]^{2-\gamma},$$

$$b_1|_{n\to\infty}\to\frac{2^{4-\gamma}\Gamma[(3-\gamma)/2]}{\Gamma[(4-\gamma)/2]}n, \tag{A10}$$

$$b_2=\sum_k[2\sin(\varphi_k/2)]^{4-\gamma},$$

$$b_2|_{n\to\infty}\to\frac{2^{6-\gamma}\Gamma[(5-\gamma)/2]}{\Gamma[(6-\gamma)/2]}n. \tag{A11}$$

As n goes to ∞ (and γ being not too closed to 2^-) a_1 goes to zero as $n^{\gamma-2}$: If to replace $b_{1,2}$ in the definition of a_1 by their asymptotic values (A10) and (A11), a remarkable cancellation, that gets rid of the linear over n term in a_1, takes place. The cancellation occurs as a direct consequence of the incompressibility of the flow (the divergenceless of the $\mathcal{K}$ matrix). Thus the major contribution into a_1 is estimated by the first term of the series over n,

$$a_1|_{n\to\infty}\to n^{-(2-\gamma)/(d-1)}, \tag{A12}$$

where it is calculated that the angular size of the elementary cell, which appeared after sectioning of the S_d sphere into n parts, is proportional to $\pi/n^{1/(d-1)}$. Considering S_2 geometry in higher dimensions $d>2$, and generally S_l geometry in $d>l$, one finds a linear growth of a_1 as n goes to ∞.

2. S_2 geometry: The general two-dimensional case

Let us fix the initial and final vectors that define the symmetric geometry [say, $\boldsymbol{\rho}_1(0)$ and $\boldsymbol{\rho}_1(T)$] to be nonparallel to each other. The situation starts to be more complicated, if one is going to solve the same instanton equation (3.1), as new parameters describing rotation of the vector come into the game. We consider here the two-dimensional case, which is the simplest one (rotation is Abelian in this case). In $d=2$ the vectors $\boldsymbol{\rho}_i$ and $\mathbf{p}_i$ can be parametrized as $\rho_i^\alpha\to\rho\,\exp[i(\varphi+\varphi_k)]$, $p_i^\alpha\to p\,\exp[i(\psi+\varphi+\varphi_k)]$, where scalars $p(t)$, $\rho(t)$ and angles $\varphi(t)$, $\psi(t)$ are time dependent. Four equations describing the dynamical behavior are

$$\dot{p}+Dp^2\rho^{1-\gamma}\left[[3-\gamma\cos^2(\psi)]b_1+\frac{\gamma-4}{4}b_2\right]\frac{\cos(\psi)}{2}=0, \tag{A13}$$

$$\dot{\psi}+\dot{\varphi}+Dp\rho^{1-\gamma}\left[[\gamma\cos^2(\psi)-1]b_1+\frac{4-\gamma}{4}b_2\right]\frac{\sin(\psi)}{2}=0, \tag{A14}$$

$$\dot{\rho}=\frac{Dp\rho^{2-\gamma}}{2-\gamma}\left[(3-\gamma)b_1+\frac{\gamma-4}{4}b_2\right]\cos(\psi)-2\kappa p\ \cos(\psi), \tag{A15}$$

$$\dot{\varphi}=\frac{Dp\rho^{1-\gamma}}{2-\gamma}\left[b_1+\frac{\gamma-4}{4}b_2\right]\sin(\psi)-2\kappa\frac{p}{\rho}\sin(\psi), \tag{A16}$$

where b_1 and b_2 were introduced in Eqs. (A10) and (A11). The equations of motion (A13)–(A16) are compatible with the conservation laws of energy (3.11) and the angular momentum (3.12) (that is pseudoscalar in this case). The system of equations (A13)–(A16) can be analyzed in full glory. Nevertheless, starting from the point one drops the diffusion term responsible for the ultraviolet regularization only. Then in the diffusion-free case, the system of equations (A13)–(A16) being rewritten in terms of the auxiliary x,y variables, $(x,y)=\rho^{\gamma/2}(\cos[\gamma\varphi/(2\sqrt{\eta})],\sin[\gamma\varphi/(2\sqrt{\eta})])$, describes a uniform motion of a particle with constant speed in the x-y plane:

$$x(t)=r'^{\gamma/2}+\frac{t}{T}\{\cos[\gamma\varphi_*/(2\sqrt{\eta})]\sqrt{R^{\gamma}-2R^{\gamma/2}r'^{\gamma/2}\cos[\gamma\varphi_*/(2\sqrt{\eta})]+r'^{\gamma}}-r'^{\gamma/2}\}, \tag{A17}$$

$$y(t)=\frac{t}{T}\sin[\gamma\varphi_*/(2\sqrt{\eta})]\sqrt{R^{\gamma}-2R^{\gamma/2}r'^{\gamma/2}\cos[\gamma\varphi_*/(2\sqrt{\eta})]+r'^{\gamma}}, \tag{A18}$$

where

$$\eta=\frac{[(4-\gamma)b_2/4-b_1]}{[(4-\gamma)b_2/4-(3-\gamma)b_1]}, \tag{A19}$$

and the following initial and final conditions for ρ,φ dynamical fields are imposed: $\rho(0)=r'$, $\rho(T)=R$ and $\varphi(0)=0$, $\varphi(T)=\varphi_*$. The solution (A17) and (A18) means, particularly, that there are no equivalent trajectories for φ_* taken from the interval $|\varphi_*|\leqslant 2\pi\sqrt{\eta}/\gamma$. Thus, considering the final values of φ_* and $\varphi_*+2\pi n$ from the interval to be equivalent, one observes $\sqrt{\eta}/\gamma$-fold degeneracy (all those trajectories from the degenerate set differ in the values of energy E and angular momentum M).

The action $\mathcal{S}$ on the ''classical'' trajectory gets the following form:

$$\mathcal{S}^{\text{cl}}=\frac{4(2-\gamma)n}{\gamma^2[(4-\gamma)b_2/4-(3-\gamma)b_1]D}\,\frac{R^{\gamma}-2r'^{\gamma/2}R^{\gamma/2}\cos[\gamma\varphi_*/(2\sqrt{\eta})]+r'^{\gamma}}{T}. \tag{A20}$$

The momentum of the $k=0$ particle at zero moment of time [the object entered the self-consistency condition (3.6) and the preexponent term] has the following dependence on the initial and final conditions imposed: $\mathbf{p}_i(t=0)=p_0\exp[i\varphi(0)+\varphi_k+\psi_0]$,

$$p_0\cos(\psi_0)=-\frac{2(2-\gamma)}{\gamma[(4-\gamma)b_2/4-(3-\gamma)b_1]DT}\{\sqrt{R^{\gamma}-2r'^{\gamma/2}R^{\gamma/2}\cos[\gamma\varphi_*/(2\sqrt{\eta})]+r'^{\gamma}}\cos[\gamma\varphi_*/(2\sqrt{\eta})]-r'^{\gamma/2}\}r'^{\gamma/2-1}, \tag{A21}$$

$$p_0\sin(\psi_0)=-\frac{2(2-\gamma)}{\gamma\sqrt{\eta}[(4-\gamma)b_2/4-(3-\gamma)b_1]DT}\sqrt{R^{\gamma}-2r'^{\gamma/2}R^{\gamma/2}\cos[\gamma\varphi_*/(2\sqrt{\eta})]+r'^{\gamma}}\sin[\gamma\varphi_*/(2\sqrt{\eta})]r'^{\gamma/2-1}. \tag{A22}$$

3. Two-point geometry: $0<\gamma<1$

The saddle-point system of equations (3.8) and (3.9) has some reduction feature at $0<\gamma<1$: if we merge a group of particles in a point and choose the momenta (equal for the particles pasted together) to be parallel to the vector connecting the points at the initial moment of time, the problem gets rid of those superfluous degrees of freedom at all the latest times too—the group can be replaced by one particle. We will check the general observation for the two-point case in the present appendix.

Here we consider a geometry formed by two groups (labeled by $+$ and $-$) of particles ($n_+ + n_- = 2n$) each merged in a point with a position $\boldsymbol{\rho}_+(t)$ or $\boldsymbol{\rho}_-(t)$, respectively. Indeed, there exists a solution of the saddle-point equations (3.8) and (3.9) that preserves the symmetry dynamically. That is specific about $\gamma<1$, it is an algebraic decay of $\mathcal{K}_r^{\alpha\beta;\eta}$ when r goes down scales; one can put the particles in a point without any divergences appearing. The number of dynamical degrees of freedom is reduced from $4nd$ to $2d$. One gets

$$\dot{p}_{\pm}^{\alpha}+n_{\mp}p_{+}^{\beta}p_{-}^{\eta}\mathcal{K}_{\rho}^{\beta\eta;\alpha}=0,\quad \boldsymbol{\rho}=\boldsymbol{\rho}_+-\boldsymbol{\rho}_-, \tag{A23}$$

$$\dot{\rho}_{\pm}^{\alpha}+2\kappa n_{\pm}p_{\pm}^{\alpha}-n_{\mp}\mathcal{K}_{\rho}^{\alpha\beta}p_{\mp}^{\beta}=0. \tag{A24}$$

Coming from $\boldsymbol{\rho}_{\pm}, \mathbf{p}_{\pm}$ variables to collective ones $\mathbf{p}=n_+\mathbf{p}_+=-n_-\mathbf{p}_-$, $\boldsymbol{\rho}$ (the full momentum is constant), one gets the system of equations that does not depend on the numbers of $+$ and $-$ particles at all,

$$\dot{p}^{\alpha}-p^{\beta}\mathcal{K}_{\rho}^{\beta\eta;\alpha}p^{\eta}=0, \tag{A25}$$

$$\dot{\rho}^{\alpha}+4\kappa p^{\alpha}+2\mathcal{K}_{\rho}^{\alpha\beta}p^{\beta}=0. \tag{A26}$$

Notice that the rotationless variant of Eqs. (A25) and (A26) is transformed to Eqs. (A1) and (A2) and the two-dimensional variant of Eqs. (A25) and (A26) is transformed to Eqs. (A13)–(A16), if one performs a reduction $a_1\to 2(d-1)/(2-\gamma)$, and $\kappa\to 2\kappa$ and $\eta\to\gamma^2/4$ there. Thus the formulas for the classical action (A8) and the initial momentum (A9) are valid with appropriate changes [the multiplier $(2n)^{-1}$ should be accounted for additionally in the action] in the case of the two-point geometry too. One finds that the effective classical action (2.17) is n independent in this case, if the values of T and R are considered to be fixed.

To check the dynamical stability of the two-point geometry let us consider a ''dumbbell'' geometry (floated variant of the two-point one) with the characteristic size of the drops of the dumbbell being initially (at $t=0^+$) much smaller than the relative distance between them. The major question to ask is; will the dumbbell geometry be preserved dynamically? To answer the question, let us go back to the classical equation (3.9). Introduce a small excursion of a particle $\boldsymbol{\rho}_{\delta}$ from the drop's center. Then, keeping the leading (over $\delta\rho/\rho$) term in the right-hand side of Eq. (3.9) one arrives at the following equation for $\delta\boldsymbol{\rho}$:

$$\delta\dot{\rho}^{\alpha}=p\rho^{1-\gamma}[\delta\rho^{\alpha}-d(\delta\boldsymbol{\rho}\cdot\mathbf{n})n^{\alpha}]. \tag{A27}$$

Substituting the already known law of the basic two-point stretching, one gets

$$\frac{\delta\rho_{\perp}(t)}{\delta\rho_{\perp}(0^+)}=\left(\frac{r'}{\rho}\right)^{(2-\gamma)/[2(d-1)]}, \tag{A28}$$

$$\frac{\delta\rho_{\parallel}(t)}{\delta\rho_{\parallel}(0^+)}=\left(\frac{\rho}{r'}\right)^{1-\gamma/2}, \tag{A29}$$

where $\delta\rho_{\perp,\parallel}$ are transversal and longitudinal (with respect to the direction of the classical stretching) sizes of the drops. To conclude, the ratios of the drops' sizes to the distances between the drops are getting smaller with time and the two-point geometry is indeed preserved dynamically. Note that Eq. (A28) is a classical manifestation of the law of volume conservation valid at $\gamma=0$ (the respective law of area conservation allows solving the problem at $d=2$ explicitly [12,18]): $\rho(\delta\rho_{\perp})^{d-1}$ does not depend on time. In other words, Eq. (A28) is a statistical descendant of the volume conservation law (incompressibility condition) valid for any particular flow.

Considering the dumbbell geometry at $\gamma>1$, one finds that it is destroyed dynamically: The size of the drops and separation between them, in the initially served two-point geometry, turns out to be of the same order at the latest times.

APPENDIX B: GAUSSIAN FLUCTUATIONS ABOUT THE TWO-POINT INSTANTON

1. Relative fluctuations of the points

Consider quantum mechanics, Eqs. (4.11)–(4.14), that appears when accounting for the relative fluctuations of the drops only. There are d essential distances in the case: $\tilde{\mathbf{r}}=\tilde{\mathbf{r}}_{\{1\}}-\tilde{\mathbf{r}}_{\{2\}}$. Operator $\hat{\mathcal{L}}'$ Eq. (4.13), rewritten in terms of $\mathbf{s}$ [related to $\tilde{\mathbf{r}}$ via Eq. (4.10)], is

$$\hat{\mathcal{L}}'=\left[\frac{d+1-\gamma}{2-\gamma}\delta^{\alpha\beta}-n^{\alpha}n^{\beta}\right]\boldsymbol{\nabla}_{\mathbf{s}}^{\alpha}\boldsymbol{\nabla}_{\mathbf{s}}^{\beta}+\frac{1}{\gamma}\left[-\left(\frac{4-\gamma}{2}+\frac{2(2-\gamma)}{d-1}\right)s^{\alpha}+\frac{2(2-\gamma)d}{d-1}(\mathbf{n}\cdot\mathbf{s})n^{\alpha}\right]\boldsymbol{\nabla}_{\mathbf{s}}^{\alpha}$$
$$-\frac{(2-\gamma)^2}{\gamma^2(d-1)}\left[-\left(\frac{1}{2}+\frac{1}{d+1-\gamma}\right)s^2+\left(2-\gamma/2+\frac{1}{d+1-\gamma}\right)(\mathbf{n}\cdot\mathbf{s})^2\right], \tag{B1}$$

where $\mathbf{n}=\boldsymbol{\rho}^{\mathrm{cl}}/\rho^{\mathrm{cl}}$. It is seen clearly from Eq. (B1) that the transversal part of the effective potential is not bound from below. Accounting for nonlinear terms is required to regularize the divergence (see Sec. IV and Appendix C for an explanation of how to avoid an explicit calculation of the terms). By this means, to describe the longitudinal fluctuation one should make $\mathbf{s}$ parallel to $\mathbf{n}$ in Eq. (B1) and so deal with the reduced operator

$$\hat{\mathcal{L}}''=\frac{d-1}{2-\gamma}\frac{d^2}{ds^2}+\frac{(d+2-\gamma)(d-1)}{2-\gamma}\frac{1}{s}\frac{d}{ds}+\frac{4-3\gamma}{2\gamma}s\frac{d}{ds}$$
$$-\frac{(3-\gamma)(2-\gamma)^2}{2\gamma^2(d-1)}s^2. \tag{B2}$$

In the long time ($\tau\gg 1$) asymptotic the major contribution into Ψ_T stems from the lowest eigenstate of $\hat{\mathcal{L}}''$. The eigenfunction of the ground state is an exponential of the quadratic form

$$\Psi_{\mathrm{gr}}\sim\exp\left(-\frac{as^2}{2}\right). \tag{B3}$$

In the Heisenberg representation the eigenfunction appears as $\exp(\Delta_E\tau)\Psi_{\mathrm{gr}}$, where the ground state energy for the eigenvalue problem gets

$$\Delta_E^{\rm st}=\frac{2+d-\gamma}{2\gamma}\left(2-\frac{3}{2}\gamma+\sqrt{(2-3\gamma/2)^2+2(2-\gamma)(3-\gamma)}\right). \tag{B4}$$

2. Inner fluctuations of the points

$\mathcal{K}$, $\mathcal{A}$, and $\mathcal{B}$ responsible for the inner fluctuations of a point (disk) are estimated as

$$\mathcal{K}_\delta\sim\delta\rho^{2-\gamma},\quad \mathcal{A}_\delta\sim p\,\delta\rho^{1-\gamma},\quad \mathcal{B}_\delta\sim p^2\delta\rho^{-\gamma}, \tag{B5}$$

where $\delta\rho(t)$ is related to $\rho(t)$ via Eqs. (A28) and (A29). Thus instead of Eq. (4.10) one has to perform the following transformation to the dimensionless $\tau_\delta,\mathbf{s}_{\delta i}$ variables:

$$\tau_\delta=\frac{\gamma}{2}\int_{r'}^{\rho}\frac{d\rho}{\rho}\left(\frac{\delta\rho}{\rho}\right)^{1-\gamma},\quad \delta\mathbf{s}_{\delta i}=\frac{R^{\gamma/4}\tilde{\mathbf{r}}_i}{\sqrt{TD\,\delta\rho\rho^{1-\gamma/2}}}, \tag{B6}$$

to get rid of the temporal dependence of the Hamiltonian. The resulting quantum mechanics will yield positive spectrum with a constant gap. The dependences of Δ_E and δ_E on n are estimated as $\sim$const and $\sim 1/n$, respectively (the potential $\mathcal{K}$ term has an extra smallness $\sim 1/n^2$, while the number of elementary excitations is $\sim n$). The dimensionless time of evolution is short $\sim[\delta\rho(0^+)/\rho(0^+)]^{1-\gamma}/[1-\gamma]\ll 1$, if γ is not too close to 1. These collected observations result in the following asymptotic for $\Psi_{\delta T}$ [analog of Eq. (4.15)]: $\Psi_{\delta T}\to\delta\chi[\tilde{\mathbf{r}}(L/r')^{1-\gamma/2}]$ [we have calculated Eq. (A29) here]. The respective contribution of the fluctuations is thus scale (r) independent,

$$\mathcal{Z}_\delta^{\rm fl}\sim\int\prod d\tilde{\mathbf{r}}_i\,\frac{\exp(-\tilde{r}_i^\alpha[\hat{\mathcal{G}}^{-1}]_{ij}^{\alpha\beta}\tilde{r}_j^\beta/2)}{\sqrt{|\det[\hat{\mathcal{G}}]|}}$$

$$\times\delta\chi[\tilde{\mathbf{r}}(L/r')^{1-\gamma/2}]\sim q^{2dn},$$

$$q=\max[1;L^\gamma/(nDT)], \tag{B7}$$

where, accounting for scale dependence only, we use $\delta\chi[\tilde{\mathbf{r}}(L/r')^{1-\gamma/2}]$, as a function of $\tilde{\mathbf{r}}$, and it decays on the scale $r'^{1-\gamma/2}L^{\gamma/2}\sim rL^\gamma/(nDT)$.

APPENDIX C: INTEGRATION OVER THE SOFT ROTATION MODE

The measure of functional integration in Eq. (2.10) is quasi-invariant with respect to a slight rotation of the fields,

$$\boldsymbol{\rho}_i(t)\to\hat{U}^{-1}(t)\boldsymbol{\rho}_i(t),\quad \mathbf{p}_i(t)\to\hat{U}^{-1}(t)\boldsymbol{\rho}_i(t),$$

$$\det[\hat{U}(t)]=1,\quad \hat{U}(T)=\hat{U}(0)=\hat{1}, \tag{C1}$$

where $\hat{U}(t)$ is a unitary matrix realizing the $d\times d$ representation of SU(d). Quasi-invariance means that an extra term appears at the transformation of the action (2.11)

$$\Delta\mathcal{S}=\int_0^T[\hat{U}[\hat{U}^{-1}]']^{\alpha\beta}\rho_i^\alpha p_i^\beta dt. \tag{C2}$$

To fix the gauge quasi-invariance we will use a general method [28] that is popularly known in field theory. To integrate over the soft mode we will put under the functional integration (2.10) the unity

$$1=\int\mathcal{D}\hat{U}(t)\,\delta([(\hat{U}\boldsymbol{\rho}_1)'\times\hat{U}\boldsymbol{\rho}_1]/\rho_1^2)$$

$$\times\det\{\delta[(\hat{U}\boldsymbol{\rho}_1)'\times\hat{U}\boldsymbol{\rho}_1/\rho_1^2]/\delta\hat{U}\}, \tag{C3}$$

where $\mathbf{n}_1=\boldsymbol{\rho}_1/\rho_1$ and $\times$ stands for the antisymmetric vector product of d-dimensional vectors. Thus, we use a requirement for one of the particles (labeled by 1), to not rotate over the origin, as a gauge condition (without loss of generality one can choose an arbitrary direction, characterized by the trajectory, to be nonrotating). For the sake of simplicity, let us consider the two-dimensional version of Eq. (C3),

$$1=\int\mathcal{D}\varphi(t)\,\delta([(\hat{U}\boldsymbol{\rho}_1)'\times\hat{U}\boldsymbol{\rho}_1]/\rho_1^2)$$

$$\times\det\{\delta[(\hat{U}\boldsymbol{\rho}_1)'\times\hat{U}\boldsymbol{\rho}_1/\rho_1^2]/\delta\varphi\}, \tag{C4}$$

$$\hat{U}(t)=\begin{pmatrix}\cos\varphi & \sin\varphi\\ -\sin\varphi & \cos\varphi\end{pmatrix}.$$

Direct calculation gives

$$(\hat{U}\boldsymbol{\rho}_1)'\times\hat{U}\boldsymbol{\rho}_1/\rho_1^2=\dot{\varphi}+(\dot{\boldsymbol{\rho}}_1\times\boldsymbol{\rho}_1)/\rho_1^2. \tag{C5}$$

Thus the determinant from the right-hand side of Eq. (C4) does not depend on the dynamical field $\boldsymbol{\rho}_1$, and we can drop the determinant. As a next step let us perform the change of variables (C1) in the original functional integral (2.11) [with the unity (C4) substituted into the integrand]. The Jacobian of such a transformation is unity. The δ function from the right-hand side of Eq. (C4), describing the gauge condition, turns out to be φ independent. We get finally that the only factor calculated integration over the soft (gauge) mode is

$$\int\mathcal{D}\varphi\,\exp[-\Delta S\{\varphi\}],\quad \Delta S=\int_0^T\dot{\varphi}\varepsilon^{\alpha\beta}\rho_i^\alpha p_j^\beta dt, \tag{C6}$$

where $\hat{\varepsilon}$ is the antisymmetric 2×2 tensor. Further, the integration over $\mathcal{D}\varphi$, being performed, reduces Eq. (C6) to the δ function

$$\delta([\varepsilon^{\alpha\beta}\rho_i^\alpha p_i^\beta]'). \tag{C7}$$

The condition under the δ function, which is satisfied on a saddle-point solution as describing conservation of the angular momentum (3.12), is thus valid for fluctuations too.

[1] R. H. Kraichnan, Phys. Fluids **11**, 945 (1968).
[2] R. H. Kraichnan, Phys. Rev. Lett. **72**, 1016 (1994).
[3] R. H. Kraichnan, V. Yakhot, and S. Chen, Phys. Rev. Lett. **75**, 240 (1995).
[4] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **52**, 4924 (1995).
[5] K. Gawędzki and A. Kupiainen, Phys. Rev. Lett. **75**, 3608 (1995).
[6] M. Chertkov and G. Falkovich, Phys. Rev. Lett. **76**, 2706 (1996).
[7] D. Bernard, K. Gawędzki and A. Kupiainen, Phys. Rev. E **54**, 2564 (1996).
[8] L. N. Lipatov, Zh. Éksp. Teor. Fiz. **72**, 411 (1977) [Sov. Phys. JETP **45**, 216 (1977)].
[9] B. I. Shraiman and E. D. Siggia, Phys. Rev. E **49**, 2912 (1994).
[10] M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. E **51**, 3974 (1995).
[11] M. Chertkov, A. Gamba, and I. Kolokolov, Phys. Lett. A **192**, 435 (1994).
[12] E. Balkovsky, M. Chertkov, I. Kolokolov, and V. Lebedev, Pis'ma Zh. Éksp. Teor. Fiz. **61**, 1012 (1995) [JETP Lett. **61**, 1049 (1995)].
[13] G. Falkovich, I. Kolokolov, V. Lebedev, and A. Migdal, Phys. Rev. E **54**, 4896 (1996).
[14] C. de Dominicis, J. Phys. (Paris) Colloq. **37**, C1-247 (1976).
[15] H. Janssen, Z. Phys. B **23**, 377 (1976).
[16] V. Gurarie and A. Migdal, Phys. Rev. E **54**, 4908 (1996).
[17] E. Balkovsky, G. Falkovich, I. Kolokolov, and V. Lebedev, Phys. Rev. Lett. (to be published).
[18] B. Shraiman and E. Siggia, C.R. Acad. Sci. **321**, 279 (1995).
[19] U. Frisch, *Turbulence. The Legacy of A.N. Kolmogorov* (Cambridge University Press, Cambridge, England 1995).
[20] A. Majda, J. Stat. Phys. **73**, 515 (1993).
[21] G. Falkovich, Phys. Rev. E **49**, 2468 (1994).
[22] V. Yakhot, Phys. Rev. E **55**, 329 (1997).
[23] A. Polyakov, Phys. Rev. E **52**, 6183 (1995).
[24] B. Shraiman and E. Siggia, Phys. Rev. Lett. **77**, 2463 (1996).
[25] M. Chertkov, G. Falkovich, and V. Lebedev, Phys. Rev. Lett. **76**, 3707 (1996).
[26] R. A. Antonia, E. J. Hopfinger, Y. Gagne, and F. Anselmet, Phys. Rev. A **30**, 2704 (1984).
[27] K. R. Sreenivasan, Proc. R. Soc. London, Ser. A **434**, 165 (1991).
[28] L. D. Faddeev and V. N. Popov, Phys. Lett. **25B**, 29 (1967).